Projective wellorder with the tree property

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Definition

Let M be a transitive model of set theory. We say that there is a *projective wellorder* of $H(\omega_1)$ in M if there is a formula $\varphi(v, u)$ with the free variables shown which defines a wellorder of $H(\omega_1)$:

$$M \models \text{“}\{(x, y) \mid H(\omega_1) \models \varphi(x, y)\} \text{ is a wellorder of } H(\omega_1)\text{”}.$$

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Remark. $H(\omega_1)$ contains all the reals (subsets of ω). We can often consider just a wellorder of the reals (with definability in $H(\omega_1)$ or in the second-order arithmetics).

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Notice that CH holds in L, and hence in L:

$$(*) \quad |H(\omega_1)| = \omega_1.$$

Question. Is an L-like model, or at least $(*)$, essential for projective definability?

The answer is negative: for instance Harrington showed already in 1976 that it is consistent that 2^ω is arbitrarily large and there is a lightface projective (in fact Δ^1_3) wellorder of the reals.

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Aims. One of the important criteria for "fatness" is an amount of compactness which is exhibited by small cardinals such as $\aleph_2$, in particular if they imply $\neg CH$ (and thus $V \neq L$). Another criterion is the forcing axioms.

We consider the following result and discuss methods for its solution (an ongoing work).

Tentative theorem (Friedman, H., Stejskalova (2019))

It is consistent from a weakly compact cardinal that $2^\omega = \omega_2$, there is a Δ^1_3 welloder of reals, and the tree property holds at ω_2.[a] *If so required, Martin's Axiom can hold as well.*

[a] Possibly some other favorite compactness principle. The tree property at ω_2 prohibits the existence of ω_2-Aronszajn trees.

Let us discuss how to achieve a weaker result:

Claim (Friedman, H., Stejskalova (2019))

It is consistent from a weakly compact cardinal that $2^\omega = \omega_2$, there is a wellorder of $H(\omega_2)$ lightfact definable in $H(\omega_2)$, and the tree property holds at ω_2. If required Martin's axiom can hold as well.

Definability in $H(\omega_2)$ allows, in particular, quantification over subsets of ω_1.

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1. Start with a nice model, such as L, and fix a long enough locally L-definable sequence $\vec{S} = \langle S_\alpha \mid \alpha < \mu \rangle$ of objects which will be used to code the wellorder. These objects should be hard to kill unless one uses a specific well-picked forcing (stationary sets, Souslin trees, etc.).

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2. Design an iteration which will add new reals while coding their order by means of selective kills of the elements S_α in $\vec{S}$.
3. Read off the wellorder from the objects in $\vec{S}$ – note that L is lightface definable in any transitive model, and so is $\vec{S}$ – checking whether the relevant S_α's have been killed or not (for instance whether they are still stationary in the generic extension).

Example. Let $\vec{S} = \langle S_\alpha \,|\, \alpha < \omega_2 \rangle$ be an $H(\omega_2)^L$-definable sequence of almost disjoint stationary/ costationary subsets of ω_1. S_α cannot be killed (S_α is killed if its complement contains a club in the generic extension) by any proper forcing, but can be killed by a relatively well-behaved non-collapsing ω_1-distributive forcing. Such an $\vec{S}$ can be used to code a wellorder of up to ω_2-many reals in $H(\omega_2)$: to read off the wellorder we need to quantify over elements of $\vec{S}$.

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Remark. An extra localization forcing (and some refinement of the method) is needed to express this wellorder in $H(\omega_1)$. We will omit this here.

Recall our goal:

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Problem. The general method described above cannot be applied to prove the Claim because ω_2 having the tree property implies that $\omega_2 = \kappa$ is weakly compact in L. In particular there are not sufficiently many stationary subsets of ω_1 (or of any fixed $\lambda < \kappa$) in L to code the wellorder of κ-many reals.

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- We will use a suitable sequence $\vec{S} = \langle S_\alpha \mid \alpha < \kappa \rangle$ of stationary/costationary subsets of $\beta^+ \cap \mathrm{cof}(\omega)$, where $\beta < \kappa$ ranges over inaccessible cardinals.
- $\vec{S}$ is lightface definable in $H(\kappa)^L$.
- The elements of $\vec{S}$ can be selectively killed without collapsing cardinals (and without coding wrong information) by shooting clubs through intervals (β, β^+), $\beta < \kappa$ inaccessible. This uses the mutual stationarity/costationarity of the S_α's: a notion introduced by Foreman and Magidor in 2001 (Acta Math.); by a result in their paper, any sequence of stationary subsets of cof ω ordinals is mutually stationary.

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  - At stage α guesses (by using a Π^1_1-diamond) initial segments T_α of a hypothetical $\kappa = \omega_2$-Aronszajn tree T in $L[G]$, adds new reals, collapses the current $\omega_2 = \alpha$ to ω_1 and specializes T_α (an argument reminiscent of the argument that PFA implies the tree property).

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    - If so required, $\mathbb{P}$ ensures Martin's axiom in $L[G]$.

Localization forcing needs to be integrated with $\mathbb{P}$ to achieve the tentative theorem discussed above:

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2. Is it possible to push this result, with suitable modifications, up one cardinal: to start with, to have the tree property at ω_3 with the wellorder of $H(\omega_3)$ definable in $H(\omega_3)$? This seems hard because we are leaving the realm of proper (or S-proper) iterations.

Thank you for your attention.